SPOT IT!® SOLITAIRE

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Abstract. The card game of Spot it® is based on an order 7 finite projective plane. This article presents a solitaire challenge: from the deck, collect a set of cards representing an order 7 affine plane and arrange those 49 cards into a square such that the symmetries of the affine and projective planes are obvious. The objective is not to simply create such a deck already in this solved position. Rather, it is to solve the inverse problem of arranging the cards of such a deck which has already been created and shuffled.

1. Introduction

As I was shopping online one day, an advertisement for Spot it!® caught my eye. This game is played with 55 circular cards. Each card has several images[1], and each pair of cards has exactly one common image (Figure 1). Several games can be played with the deck, all involving multiple players trying to be the first to spot the matching image between two cards. With my curiousity piqued, I purchased the deck, which is made by *Blue Orange Games*. I quickly discovered that the deck is two cards shy of fully representing an order 7 finite projective plane. It seemed a natural course of action to create the two missing cards and then proceed to arrange the cards into a configuration which would make it easy to demonstrate the order 7 finite projective plane. I didn't realize how fun and challenging this would be. I'm hoping the "rules" (and solution) of this single-player challenge will be entertaining to mathematicians and game-lovers alike.

2. Background

Before discussing the higher order finite projective planes and affine planes, let us address the order 3 case (Figure 2). In both the projective plane and the affine plane, points are connected by "lines", and any lines not sharing a point are parallel. In the affine plane, there are four sets of three parallel lines (which share a color), each with three points. However, in the projective plane, there are four additional points and no parallel lines! An affine plane of prime order n contains n^2 points and has $n + 1$ sets of n parallel lines, each with n points. The associated projective plane contains $n^2 + n + 1$ points and $n^2 + n + 1$ lines, each pair of points sharing a line and each pair of lines sharing a point. From the projective plane, *any* $(n+1)$ (here, four) colinear points may be removed, along with all its incident lines, and an affine plane results.

[1]Thanks to Theirry Denoual, co-founder of *Blue Orange Games*, for permission to use this artwork!

Figure 1. These two cards have an image in common.

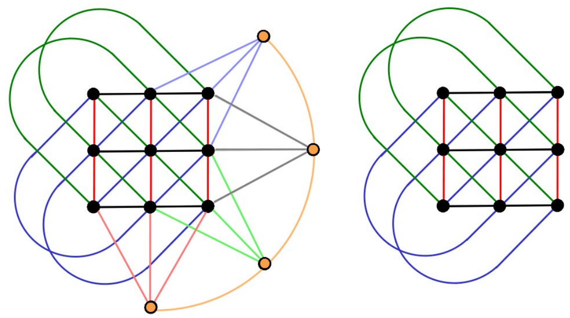

FIGURE 2. The order 3 projective (left) and affine (right) planes.

A great overview of the finite geometry behind Spot it® by Maxime Bourrigan is [1]. For those whose interest in finite geometry is beyond the scope of this discussion, I recommend [2] for those who have not yet mastered abstract algebra, and [3] and [4] for those who have. This technique generalizes for other orders, although its utility as a game diminishes due to quadratic growth of set size! This discussion is arranged so that those wishing fewer clues can read fewer sections, thus leaving more of the fun for themselves.

3. Finding the missing cards

I will presume for the moment, that the reader wishes to find the two "missing" cards in a Spot it® deck. First, list the images in the deck. To do this, locate an image which occurs 8 times (that is, $n+1$), and pull out all those cards. For example, if the deck has 8 spiders, pull out the spider "set". There can be no other commonalities in any of those 8 cards. Thus, there must be $8 \times 7 + 1$ or all 57 images on those 8 cards. Next, tabulate the frequencies for each image. One image which is present only 6 times, while 14 images are present 7 times. All other images should be present 8 times, as they are not missing from any card. The image which is missing twice must be assigned to both missing cards. Without loss of generality, assign one additional missing image to one of the missing cards. Call this image the "reference image". For the remaining 13 images, search the entire deck to see if it occurs with the reference image or not. If it does, it cannot do so again, so it must be assigned to the other missing card. If, however, it does not occur with the reference image, it should! So, it should go on the missing card which has the reference image!

4. The rules of the game

Begin by removing all cards having a specific common image which we will call the "infinity image". The remaining cards form an affine plane of the order 7 (or n). (In my Spot it® deck, the two missing cards both contain a snowman. So, if I simply remove all the snowman cards at this step, I do not need to actually find the two missing cards in order to proceed.) The infinity cards are analogous to the orange points in Figure 2.

The ultimate goal is to lay out the remaining 49 (or n^2) cards in a 7 by 7 (or n by n) grid so that this one rule is satisfied: Pick any two cards in the grid. Let their positions in the grid be (x, y) and $(x+h, y+k)$ with x and y numbered between 0 and 6 inclusive. The common image found on these two cards must also be at position $(x+2h \mod 7, y+2k \mod 7)$ (or $(x+2h \mod n, y+2k \mod n)$). For example, in Figure 3, the solved $n=5$ case is shown. Consider the (row, column) positions (1,1) and (2,4). The red colored circle (with a plus symbol) is common to these cards, so at (3,2), we expect to find this symbol again, and it is there! Since 7 (or n) is prime, and all elements are generators in $\mathbb{Z}_7$ (or $\mathbb{Z}_n$), there will be 7 (or n) such images in a set, within the 7 by 7 grid. This also implies that each row (and each column) will have a common image. (The symbols in Figure 3 give the same information as the colors. For example, all the red symbols have a "plus" sign on them.)

The families of parallel images in the $n=5$ affine plane are analogous to the parallel lines in Figure 2. For one parallel family, all images lie on lines having slope 1. In Figure 3, this is the colored triangle family.

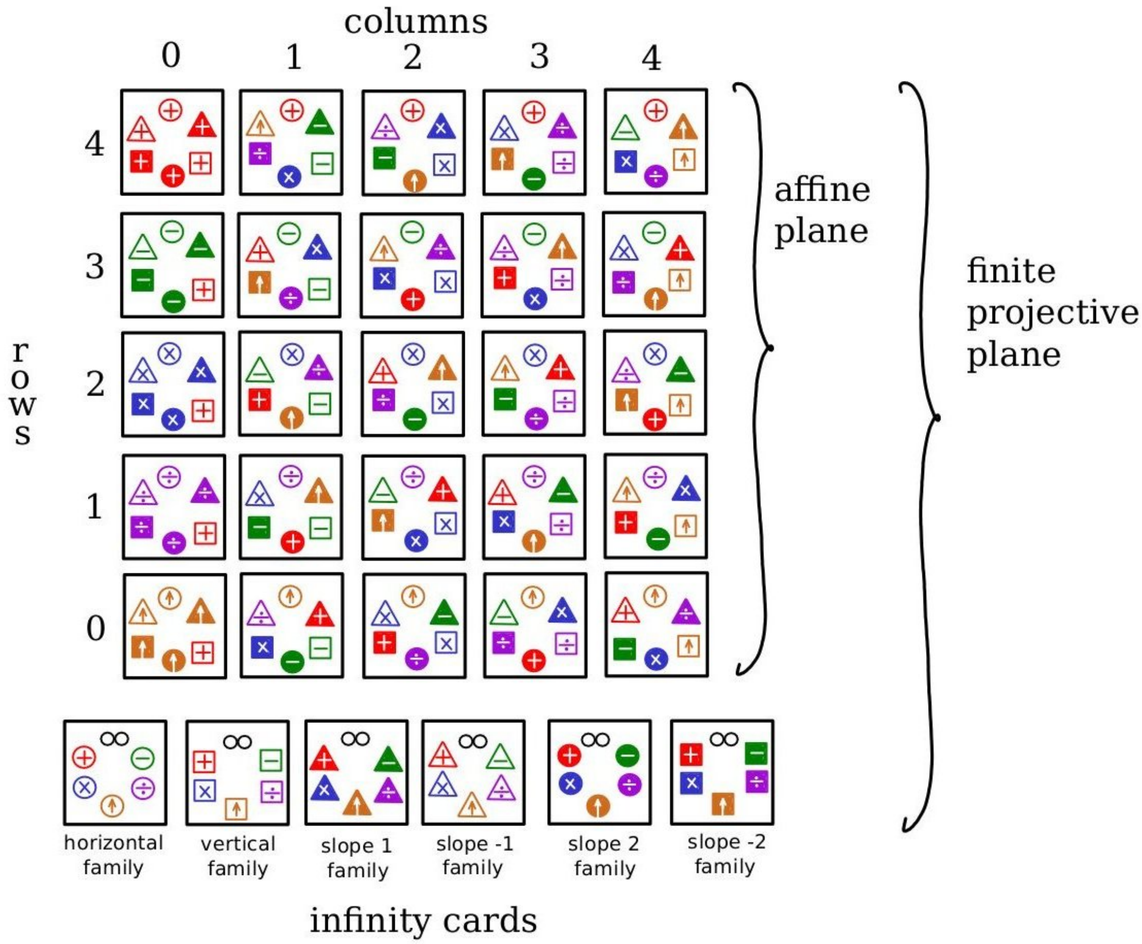

FIGURE 3. The solved challenge for the $n = 5$ case.

The purple colored triangles lie on the line $r = c + 1 \mod 5$, while the red ones lie on the line $r = c + 4 \mod 5$. The blue colored squares are from another parallel family, and they lie on the line $r = -2c + 2 \mod 5$. (Another equivalent equation for this line is $c = 1 + 2r \mod 5$.)

The method for creating such a deck of cards should be obvious now. First, generate the affine grid using parallel lines. Then, each parallel image family is placed on a new card, together with the infinity image. However, the goal here is not to create such a deck but rather to properly arrange an already existing deck! Those readers desiring the maximal fun should now attempt to solve this inverse problem without reading further! One more warning will be given after the easy clues are presented.

5. The initial setup of the grid

First pull out any set of 8 cards which share an image, to be used as the infinity cards. (Or, if the missing cards have not been created, the 6 cards sharing the twice-missing image should be pulled aside.) Next, select one of the infinity cards to represent your row (horizontal) family and one to represent your column (vertical) family, and keep them in view. Arrange your grid so that each column contains a common image and each row contains a common image. (Note that there are $57 \times 8 \times 7 \times 7! \times 7!$ ways to make these choices. There are 57 images to pick as the infinity image, then 8 cards from that infinity set which can be used to define the rows. Once the rows are chosen, 7 cards remain to define the columns. There are 7! ways to order rows and 7! ways to order columns. Also, note that the $n = 3$ case is fully solved at this stage.) Next, by

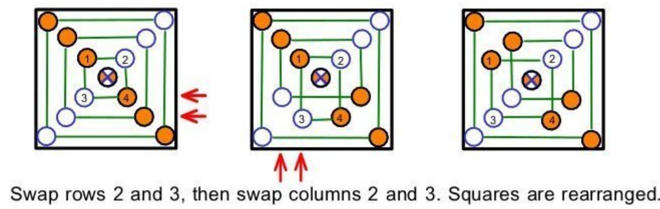

FIGURE 4. Squares on the diagonal deforming and returning again

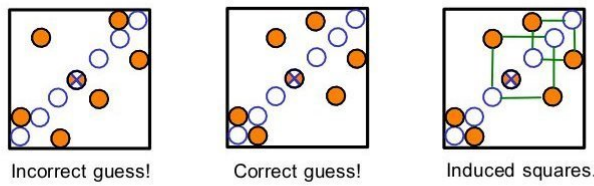

FIGURE 5. Choosing the correct image for the second diagonal

swapping rows and/or columns of cards, place a common image on the diagonal line $r = c$, ignoring all other parallel lines in this family with slope 1. (This is the line which runs from the lower left corner to the upper right corner.) All "moves" henceforth will consist of swapping two rows or swapping two columns. We know from abstract algebra that all permutations can be formed by swaps, so swapping rows/columns is sufficient to solve this puzzle.

The next objective is to get a common image on the second diagonal line, $r = -c-1 \mod n$ (or $r \equiv -c-1$ for brevity), ignoring all other parallel lines in this family with slope -1. (This is the line which runs from the upper left corner to the lower right corner.) As the final easy clue, you may now swap rows as you wish, so long as you also swap the columns having the same indices. For example, if rows 2 and 3 are swapped, then columns 2 and 3 (or vice versa), the net effect on the line $r = c$ is to swap the card at (2,2) with the card at (3,3). Rows and columns remain intact but are relocated. This technique allows us to maintain the $r = c$ diagonal while still giving enough freedom to finish the puzzle. Without loss of generality, freeze the middle card. Thus, you will not move the middle row or middle column. (The fact that we are working on a torus allows us to freeze any one card, but the choice of the middle card makes the solution easier to execute, due to symmetry.) This gives $6! = 720$ remaining grid arrangements, 6 of which are valid solutions. At this point, if you wish to have any fun with the puzzle, you should stop reading. This is your last warning.

6. Finding the second diagonal and finishing the puzzle

The somewhat surprising fact is that once you have chosen the image for the line $r = c$ and have frozen the middle card, the image for the line $r \equiv -c - 1$ is already determined! It has to be one of the images on your middle card, but it cannot be the image of that row or of that column or of the line $r = c$. So, there appear to be 5 options remaining. But that is not so! Only one will work, and any attempt to set the incorrect image will end in frustration! So, how do we figure out which image will work?

Since all moves are reversible, we may simply track the scrambling process to see where images on the second diagonal $r \equiv -c-1$ may go when we allow paired swaps of rows/columns. We imagine 3 (that is, $\frac{n-1}{2}$) concentric squares around the frozen middle card to help us keep track of where the second diagonal set can go. Figure 4 demonstrates one of the 15 possible deformations of the second diagonal and also demonstrates how, as promised, the cards along $r = c$ are, as a set, invariant. The white circles represent the cards on the line $r = c$, while the orange circles represent those which will ultimately be on the line $r \equiv -c - 1$. Suprisingly, the (group) actions of swapping matched rows/columns maintains these three sets of four cards

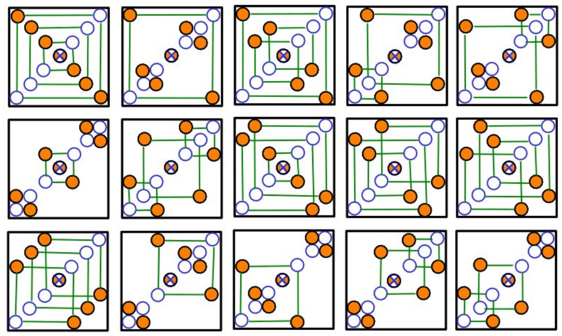

Figure 6. The 15 ways the cards of the second diagonal may initially appear.

as corners of squares which are symmetric about the line $r = c$. (However, they are not concentric except when the two diagonals are both correctly set.) To determine which of the 5 candidate images should be used as the image for the line $r \equiv -c - 1$, we simply search for the image which is already symmetric with respect to the line $r = c$. (See Figure 5.) We can think of the 6 non-fixed cards along $r = c$ as being in three pairs, thus inducing the squares. Combinatorially, this can can occur 15 ways, as shown in in Figure 6. When you get to this point in the puzzle, it is reassuring to find this symmetry, asserting that the solution is near! Simply swap rows and corresponding columns until the images on the second diagonal are set! (Note: for the order 5 case, even the incorrect candidate images for the second diagonal are arranged symmetrically with respect to $r = c$. Look for 4 misplaced candidate images instead of 2.)

7. Final moves

Once the two diagonals are set, remaining moves must not only have paired row/column moves, but must also maintain symmetry between right and left (as well as up and down). For example, if rows/columns 2 and 4 are swapped, this is already balanced and will not disrupt either diagonal line. However, if columns 0 and 1 are swapped, columns 5 and 6 must also be swapped (as well as rows 0 and 1, and also rows 5 and 6). By tightening these orbits, we close in on one of the 6 solutions.

Once the two diagonals are established, there are still 48 possible card arragements. Each "square" may be in each of the 3 locations ($3! = 6$), and each has two legal orientations as it is legal to rotate it $180°$ degrees. Since $2^3 = 8$ and $6 \times 8 = 48$, there are 48 possible arrangements of the cards, 6 of which are solutions. For example, you have the freedom to choose any one square's location and orientation, but the rest is then predetermined. For simplicity, we presume the innermost square is set properly, that is, the 9 cards in the middle of the grid are now fixed.

Now, using what is known about the families of parallel images, move the cards into their final positions. For example, since we now know which family of images has a slope of -1, we can deduce which of those images should appear on the cards in positions (0,4) and (1,3) just by looking at the card in position (2,2). If an image is not in the desired location, look for it on the opposite side of the affine grid, relative to a $180°$ rotation about the center. You might also need to swap the middle and outermost squares. In a few moves, you will see before you, a perfectly arranged affine plane.

References

[1] Maxime Bourrigan, *Dobble et la géométrie finie*. Images des Mathématiques, CNRS, 2011.
[2] Harold L. Dorwart, *The Geometry of Incidence*. Prentice-Hall, Inc, Englewood Cliffs, N.J, 1966.
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[4] Daniel R. Hughes and Fred C. Piper, *Projective Planes*. Springer-Verlag, New York, 1973.

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